M.Math. IInd year
Midsemestral exam
IInd semester 2006
Commutative Algebra — B.Sury
Maximum marks 90

All questions carry equal marks. Be BRIEF.

Q 1.
Let A be a domain in which every finitely generated ideal is principal. Prove that a A-module M is flat if and only if it is torsion-free.

OR

Let (A, M_A) and (B, M_B) be local rings and $\theta : A \to B$ be a ring homomorphism such that $\theta(M_A) \subset M_B$. If $N \neq 0$ is a finitely generated A-module, show that $N \otimes_A B \neq 0$.
Hint : Consider $B \otimes_A A/M_A$ and $N \otimes_A A/M_A$ first.

Q 2.
If P is a finitely generated projective A-module and $\mathcal{M} \subset A$ is a maximal ideal, show that $P_{\mathcal{M}}$ is a free $A_{\mathcal{M}}$-module.

OR

Let $S \subset A$ be a multiplicative subset and M, N be A-modules. Show that the canonical homomorphism

$$S^{-1}Hom_A(M, N) \to Hom_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

is injective if M is finitely generated.

Q 3.
Given any ideal I, consider the short exact sequence

$$0 \to I \to A \to A/I \to 0.$$

Given an A-module M, write out the long exact sequence of A-modules obtained by tensoring with M. Prove that M is flat if and only if $Tor_1(M, A/I) = 0$ for every finitely generated ideal I of A.

OR

If

$$0 \to K_n \to P_{n-1} \cdots \to P_0 \to M \to 0$$

is an exact sequence of A-modules with $P_0, \cdots, P_{n-1}$ projective, and if the projective dimension of M is n prove that K_n must be projective.
Hint : You may use Schanuel's lemma.

Q 4.
If A is Noetherian, prove

$$\sqrt{(0)} = \bigcap\{P : P \in Ass(A)\}.$$

OR

If A is Noetherian, M is a finitely generated A-module, and N is any A-module, prove that

$$Ass \quad Hom_A(M, N) \subseteq Ass(N) \cap Supp(M).$$

Q 5.
In $A = K[X, Y]$, show that $(X) \cap (X, Y)^2 = (X) \cap (X^2, Y)$ are both reduced primary decompositions for $I = (X^2, XY)$. Find $Ass(A/I)$.

OR

For $A = K[X, Y]$, and $M = (X, Y)$, compute $Ass(M), Supp(M)$ and a composition series for M.

Q 6.
If A is Noetherian, show that $A[[X]]$ is a flat A-module.
Hint : Use completions.

OR

Using Artin-Rees lemma or otherwise, prove :
If A is a Noetherian ring, I is an ideal, $a \in A$ is not a zero divisor, then there exists $n_0 > 0$ so that for all $n \geq n_0$, the relation $ab \in I^n$ implies $b \in I^{n-n_0}$.